Lyapunov exponents and the merger of point-vortex clusters

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Vortex clusters in two-dimensional inviscid flow are studied by long-time integration of the point-vortex equations. We compute Lyapunov exponents and the Kolmogorov-Sinai entropy (KSE) as functions of the dimensionless centroid separation μ between two clusters, each of them containing two or four point vortices of equal strengths. It is demonstrated that the KSE of two four-vortex clusters increases rapidly if μ becomes smaller than $\mu_c \approx 3.2$, and the merger time increases faster than exponentially for $\mu > \mu_c$. This result supports the conjecture that the merger of distant continuous vorticity fields is so exceedingly slow as to be numerically unidentifiable.

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The motion of two vorticity patches separated by irrotational flow is a fundamental problem of perfect fluid dynamics in two dimensions. Although this problem has been intensively studied in the past using moment models (Melander and co-workers [1,2]), contour dynamics (Dritschel [3]), direct numerical simulation (Melander, Zabusky, and McWilliams [2]) and statistical equilibrium theories (Whitaker and Turkington [4], Robert and Sommeria [5], and Lundgren and Pointin [6]) there is still a partial contradiction between the deterministic approaches [1–3] and the statistical ones [4–6]. On the other hand, the deterministic theories predict that the vortex patches rotate around each other with constant angular velocity if the dimensionless separation

$$\mu = L/l \sim (\text{vortex distance})/(\text{vortex diameter}) \qquad (1)$$

is larger than some critical value μ_c depending on the initial shape of the patches. In the case $\mu < \mu_c$ these theories predict the merger of the vortex patches into a nearly axisymmetric state or the nonexistence of periodic solutions. On the other hand, the statistical theory predicts merger for any value of μ and a fully axisymmetric final equilibrium state. To reconcile the deterministic and the statistical approaches Whitaker and Turkington [4] conjecture that the merger ultimately occurs for arbitrary μ, while the *speed* of the merger process becomes exceedingly slow for large μ. This fact could have precluded the detection of the merger in the previous direct simulations restricted to few eddy-turnover times.

The aim of the present paper is to invoke the theory of nonlinear dynamical systems, specifically Lyapunov exponents and the Kolmogorov-Sinai entropy, which measure the average time over which the state of a system can be predicted [7], in order to characterize in more detail the speed of the mixing process in phase space as a function of μ. Ideally, one would wish to compute Lyapunov exponents from numerical solutions of the Euler equations or Navier-Stokes equations at high Reynolds number, as it was done in the work of Grappin and Léorat [8]. For the problem at hand, however, this task is quite expensive computationally. Therefore we restrict ourselves to the consideration of point vortices, arranged in two distant clusters with either two or four vortices, such as sketched in Fig. 1. Although this model is only a caricature of the continuous merger problem, it offers some significant advantages. On the one hand, numerical solutions of the point-vortex equations can be obtained with high accuracy and high speed. On the other hand, the number of dynamical degrees of freedom is fixed by the number of point vortices and the computation of Lyapunov exponents is straightforward.

We consider the Hamiltonian equations of point-vortex dynamics (see e.g., Aref [9])

$$\frac{dx_\alpha}{dt} = \frac{\partial H}{\partial y_\alpha}, \quad \frac{dy_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha} \quad (\alpha = 1, \ldots, N) , \qquad (2)$$

where

$$H = -\frac{\Gamma^2}{2\pi} \sum_{\alpha<\beta}^{N} \ln|\vec{r}_\alpha - \vec{r}_\beta| . \qquad (3)$$

These equations describe the time dependence of the positions $\vec{r}_\alpha = (x_\alpha, y_\alpha)$ of point vortices with equal constant circulation Γ.

We solve numerically the Hamiltonian Eqs. (2) and (3) by a fourth order Runge-Kutta method with adaptive step size control [10]. This step size control is complemented by a routine [11] which guarantees that the relative run time error of all integrals of motion is below 10^{-7} for a run lasting typically several hundred units of the eddy-turnaround time. The computations are performed with dimensionless quantities $\vec{r}_\alpha$, t, and $\Gamma = 2\pi$, which can be formally introduced by measuring space and time in units of l and $l^2/2\pi\Gamma$ (cf. Fig. 1). For each run we evaluate the distance $\mu(t)$ between the centroids of each vortex cluster and its maximum relative variation

$$V(T) = 2\frac{\max[\mu(t)] - \min[\mu(t)]}{\max[\mu(t)] + \min[\mu(t)]}, t \in [0,T], \quad 0 \le V \le 1 \quad (4)$$

over the total integration time. Moreover, we compute the time-dependent quantities

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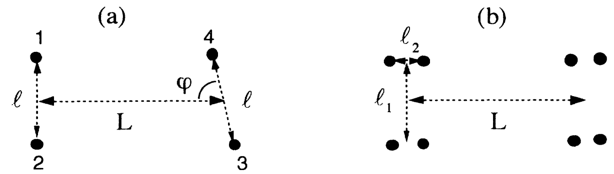

FIG. 1. Sketch of initial conditions in the four-vortex case (a) and in the eight-vortex case (b). The distances between the vortices 1-2 and 3-4 are equal in (a).

$$\lambda_i(t)=\{\ln[\varepsilon_i(t)/\varepsilon_i(0)]\}/t \quad (i=1,\ldots,2N) , \tag{5}$$

where ε_i are the distances between the unperturbed trajectory and nearby trajectories evolving from a set of orthogonal initial perturbations as detailed in the work of Benettin *et al.* [12]. For the calculation of the Lyapunov exponents, which we approximate by the values $\lambda_i(T)$ at large but finite time, we use the algorithm of Wolf *et al.* [13]. Finally, we compute the most important quantity by which chaotic motion can be characterized, namely, the Kolmogorov-Sinai entropy

$$h=\sum_{\lambda_i>0}\lambda_i(T) \tag{6}$$

as the sum of all non-negative Lyapunov exponents. It is inversely proportional to the time interval over which the state of a dynamical system can be predicted and measures the rate at which information about the state is being lost with time [7].

Let us briefly recall some basic properties of the point-vortex Eqs. (2) and (3). This system has three independent integrals of motion, resulting from the invariance properties of the two-dimensional Euler equations (Aref [9]). Therefore, a system of four point vortices is the simplest nonintegrable system, capable of displaying chaotic motion. The behavior of two point vortices with equal strengths Γ can be deduced from an analytic solution of (2). It consists of a perpetual rotation around their centroid with constant angular velocity $\Gamma/2\pi l^2$.

Consider first the motion of four point vortices as the simplest case displaying an analog of the merger phenomenon. We study the evolution starting from the initial condition sketched in Fig. 1(a). This initial condition is completely determined by the value $\mu=L/l$ of the centroid distance and by the angle φ characterizing the deviation of the initial configuration from rectangular shape. Note that for $\varphi=\pi/2$ the motion is unchanged (apart from a rescaling of coordinates and time) if μ is replaced by $1/\mu$, which serves as an additional test for the precision of the numerical computation. In the following we confine ourselves to the case $\mu>1$. In the limit $\mu\rightarrow\infty$ integrability and the existence of quasiperiodic solutions were demonstrated by Khanin [14]. Each pair rotates with angular velocity $\Gamma/2\pi l^2$ around its own centroid, while both pairs rotate around each other with angular velocity $\Gamma/\pi L^2$. Figure 2, where we plot the temporal evolution of the centroid distance for different initial values of μ, shows that for large μ the distance remains virtually constant. The weak oscillatory modulations of the upper curves indicates weak nonlinear interactions between both pairs. As the initial separation gets smaller, the centroid distance experiences vigorous fluctuations indicating an irregular vortex motion. Quantitative information about the transition from quasiperiodic to irregular motion is provided by the frequency spectra of $\mu(t)$ which are dominated by the basic frequencies and their harmonics as long as $\mu>\mu_c(\varphi)$, and which behave (quasi) continuously for $\mu<\mu_c(\varphi)$ [15]. This transition to chaotic motion coincides with a rapid increase of the fluctuation amplitude V as shown in Fig. 3(a). The high value of V indicates that the centroid distance—and thereby the position of each vortex cluster—is no longer a meaningful quantity. Visually, the particle trajectories look highly entangled in the chaotic regime and no distinction is possible between individual clusters. We can thus define the merger condition for point-vortex clusters by the value of μ below which $V(\mu)$ jumps from zero to a value close to one. For the numerical computations we use $V(\mu)=\frac{1}{2}$ as a merger condition. Figure 3(a) shows that the merger transition depends only weakly on the angle φ in the initial condition and can be expressed as $\mu_c=2.8\pm0.2$. This result can be interpreted as a signature of the existence of an almost circular region with radius $R\sim1.4l$ circumscribing the point-vortex cluster in which other point vortices are trapped once they have crossed its boundary. The existence of such a region around *isolated* point-vortex clusters has been already noted in [3]. It is, however, noteworthy that this region has physical relevance for *interacting* point-vortex clusters too. A similar phenomenon has been observed in a recent study of chaotic advection in the vicinity of individual point vortices [16].

The location of the merger transition point in the neighborhood of $\mu_c=2.8$ is not entirely fortuitous and can be partially understood from previous results of Aref and Pomphrey [17] for an integrable subproblem characterized by the initial condition $\varphi=\pi/2$. It was shown by those authors that the symmetry relation $\vec{r}_1=-\vec{r}_3$; $\vec{r}_2=-\vec{r}_4$ is an invariant of motion if it is fulfilled by the initial condition and that this particular type of evolution is completely determined by the initial value of the integral of motion

$$\Lambda^2=2^{16}\prod_{\alpha<\beta}^{4}(\vec{r}_\alpha-\vec{r}_\beta)^2\left[\sum_{\alpha<\beta}^{4}(\vec{r}_\alpha-\vec{r}_\beta)^2\right]^{-6} . \tag{7}$$

There exists a critical value $\Lambda_c=2/3\sqrt{3}$ such that if $\Lambda<\Lambda_c$, the motion of the four point vortices consists of a separate rotation of the two pairs around their common center of vorticity superposed by a separate rotation of each pair around its centroid. If $\Lambda>\Lambda_c$ the four point vortices rotate around their common center of vorticity. The condition $\Lambda=\Lambda_c$ describes the separatrix between these classes of periodic motion in terms of l and L and provides, after some algebraic manipulations, the explicit relation $\mu_c=2.876$ for the transition from separate rotation of each pair to the common rotation.

Next we turn to an analysis of the Lyapunov exponents as functions of μ and φ. In order to understand the stability properties of motion resulting from initial conditions of Fig. 1(a) we change systematically the values of μ and φ in the initial conditions. For each parameter pair (μ,φ) we calculate the complete set of eight Lyapunov

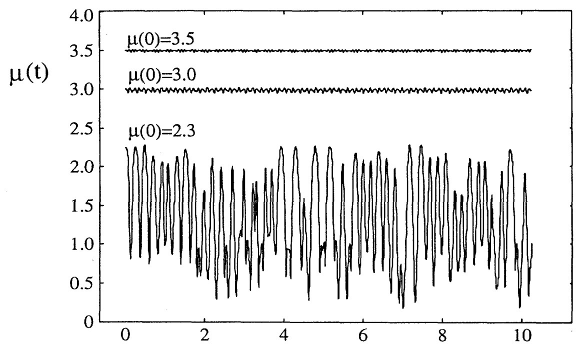

FIG. 2. Temporal evolution of the centroid distance in the four-vortex system for different initial separations of the vortex clusters and $\varphi=0$. Parameters are $l=0.2$, $\Gamma=2\pi$.

exponents. Because of the existence of three integrals of motion there can be at most one positive Lyapunov exponent, say λ_1, which determines the Kolmogorov-Sinai entropy as $h=\lambda_1$.

For large μ we find that all $\lambda_1(T)$ tend to zero as the total integration time T is increased [cf. Fig. 3(b)]. Thus, the corresponding solution for $\mu>\mu_c$ is stable in phase space. If μ decreases below a critical value μ_c, which is very close to the location of the merger condition, one Lyapunov exponent increases sharply and we observe an unstable (chaotic) trajectory. The main conclusion which can be drawn from the Kolmogorov-Sinai entropy of the four-vortex system is that the merger of clusters with two point vortices is accompanied by a transition from regular (quasiperiodic) to chaotic motion.

Consider now the motion of eight point vortices evolving from the initial condition sketched in Fig. 1(b). Notice that this is the simplest generic case in which each cluster (since comprising four vortices) is chaotic by itself. We restrict ourselves to the case $l_1=2l_2$ in which we know from the previous section that each cluster behaves chaotically by itself. We evaluate the time $T(\mu)$ after which the two clusters merge and the value of the Kolmogorov-Sinai entropy $h(\mu)$ for the resulting motion where $\mu=L/\sqrt{(l_1^2+l_2^2)^{1/2}}$.

The merger time is defined as the integration time after which V, as defined in Eq. (4), exceeds the value of $\frac{1}{2}$ for the first time. Visual observation of the point-vortex motion affirm the appropriateness of this definition. The behavior of the merging time, plotted in Fig. 4(a), can be roughly divided in two classes. For $\mu<3.2$, $T(\mu)$ is of the order of unity and shows a slow exponential increase. In the vicinity of $\mu=3.2$ the merging time jumps by approximately one order of magnitude and increases faster than exponentially for $\mu>3.2$. No merger was observed for $\mu>3.6$ within a total integration time of 10^5. The behavior of the Kolmogorov-Sinai entropy, graphed in Fig. 4(b), appears to be strongly correlated with that of $T(\mu)$. Above $\mu=3.2$, where the merging process proceeds either very slowly or does not take place at all, the Kolmogorov-Sinai entropy is almost independent of μ and, after appropriate rescaling of the time, assumes roughly twice the value of an individually moving cluster. At the critical value $\mu=3.2$ the Kolmogorov-Sinai entropy has a bend and increases with decreasing centroid separation. To test the robustness of this result, we have performed spot checks with simulations for different ratios l_1/l_2, always obtaining a bend in the KSE in the vicinity of $\mu=3.2$

The appropriateness of the Kolmogorov-Sinai entropy (KSE) for the characterization of the merger process rests on two facts. First, the KSE is an additive quantity because the total KSE of two noninteracting systems, say h, is the sum $h=h_1+h_2$ of the KSE's of each subsystem. As in classical statistical mechanics, the excess $\Delta=h-h_1-h_2$ of the total KSE over the KSE of the noninteracting subsystems can then be considered as a "mixing entropy" characterizing the degree of additional information production due to the interaction of the subsystems. Second, a peculiarity of the point-vortex model consists in the fact that the canonical coordinates of the Hamiltonian system are identical with the coordinates of the point vortices in physical space. Therefore, the Lyapunov exponents and the Kolmogorov-Sinai entropy are directly related to mixing properties of the fluid flow. Indeed, $\varepsilon_i(t)$ appearing in Eq. (5) is the sum of all distances between an unperturbed point-vortex ensemble and a perturbed ensemble, and consequently, the Kolmogorov-Sinai entropy describes the information production within one unit of time caused by all unstable infinitesimal displacements of point vortices.

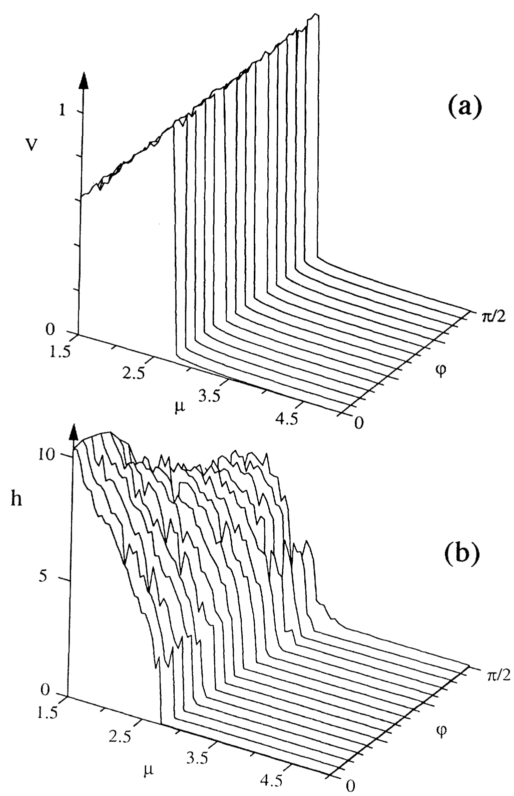

FIG. 3. Regular and chaotic motion in the four-vortex system: (a) Maximum fluctuation of the centroid distance, (b) Kolmogorov-Sinai entropy (equal to the single nonzero Lyapunov exponent) as functions of the parameters μ and φ. Parameters as in Fig. 2. Total integration time is $T=1000$. Critical values for μ are 2.75, 2.7, and 2.90 for φ equal to 0, $\pi/4$, and $\pi/2$, respectively.

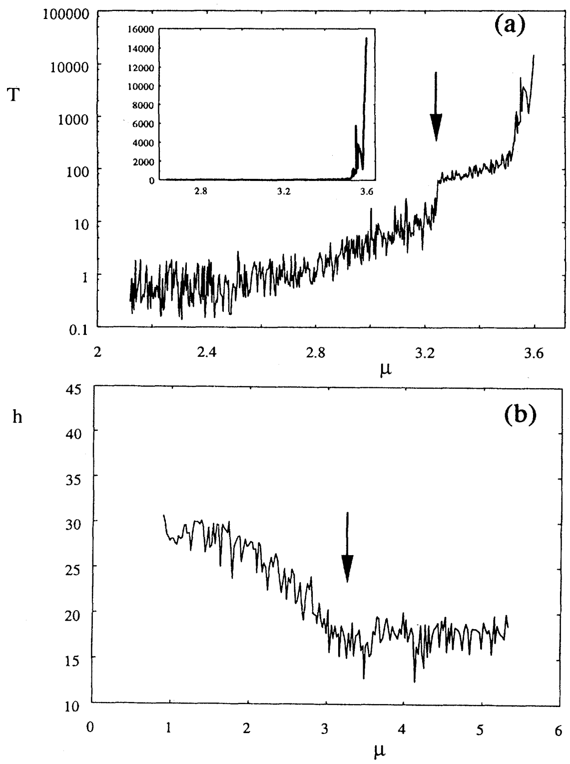

FIG. 4. Merger in the eight-vortex system: Merger time (a) and Kolmogorov-Sinai entropy (b) as a function of the initial centroid distance $\mu = L/(l_1^2 + l_2^2)^{1/2}$ for the initial condition of Fig. 1(b). Parameters are $l_1 = 0.4$, $l_2 = 0.2$, $\Gamma = 2\pi$. The inset in (a) is a linear plot of $T(\mu)$.

With these considerations in mind, we can give the following physical interpretation for the bend of the Kolmogorov-Sinai entropy as a function of μ and its correlation with the merger time. For distant clusters the mixing entropy $\Delta = h\text{-}2h_0$, where h_0 is the KSE of a four-vortex cluster, is very small indicating that mixing between the clusters is virtually absent, in agreement with the behavior of the merger time. As μ decreases below $\mu = 3.2$, the mixing entropy starts to increase, implying that the mixing in phase space (which in our case is identical with physical space) is more efficient than the mixing which would have been provided by the sum of the noninteracting subsystems.

In summary, we have presented numerical evidence for the existence of a merger phenomenon of distant point-vortex clusters. In the eight-vortex system, which represents the simplest generic case, merger occurs within few eddy-turnaround times if the separation ratio μ is smaller than 3.2. For $\mu > 3.2$ the merger time $T(\mu)$ grows faster than exponentially suggesting a singularity of $T(\mu)$ for $\mu > 3.6$. The transition at $\mu = 3.2$ is accompanied by a bend in the Kolmogorov-Sinai entropy. Our results are in qualitative agreement with the analysis of Lundgren and Pointin [6] in that the merger time is a monotonically increasing function of μ. Note, however, that the previous theory, in our notation, predicts the merger velocity $d\ln(\mu)/dt$ to be proportional to μ^{-4} [cf. Eq. (71) in [6]), while our numerical results suggest the existence of a critical μ above which this velocity decreases faster than exponentially.

One should be wary of drawing conclusions about the behavior of the infinite-dimensional Euler equation from the results of low-dimensional point-vortex models. Nevertheless, some of the present results may serve as a guideline for future studies of the continuous merger problem. In particular, it would be interesting to investigate the Kolmogorov-Sinai entropy in the continuous problem by direct simulations, similar to those of Grappin and Léorat [8], in order to verify the discontinuity of the KSE and to explore possible relations to the concept of dynamical phase transitions. A systematic study of the merging time for a few selected values of μ using direct simulation, contour dynamics or large numbers of point vortices could provide insight into the question about the existence of a singularity of $T(\mu)$ for finite separation ratios. This would imply the existence of initial conditions which do not evolve to an axisymmetric state, even if the latter is a statistical equilibrium of the two-dimensional Euler equations.

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